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On a recent result of Cazenave, Dickstein and Weissler

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ABSTRACT

In this note, we present an estimate from below of the blow-up time of blowing-up solutions of a non-linear non-local in the time evolution equation recently introduced by Cazenave et al. (2008) [1]. Moreover, we give an alternative proof of one of the results of Cazenave et al. (2008) [1] which gives more precise information on the hypotheses than those in the paper Cazenave et al. (2008) [1].

1. Introduction

The aim of this note is to establish a lower bound for the time of the blow-up for any blowing-up solution of the non-local in the space and time evolution equation

$$\begin{cases} u_t + (-\Delta)^{\frac{\beta}{2}} u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} u^p \mathrm{d}s, \quad p > 1 \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

posed in $Q := (0,\infty) \times \mathbb{R}^N$, where $0 < \beta \le 2$, $(-\Delta)^\gamma$ is the non-local operator defining the fractional power of $(-\Delta)$ in the x variable defined via the Fourier transform $\mathfrak{F}$ and its inverse $\mathfrak{F}^{-1}$ by $(-\Delta)^\beta w(x,t) = \mathfrak{F}^{-1}(|\xi|^{2\beta}\mathfrak{F}(w)(\xi))(x,t)$, and $u \ge 0$.

Our article is motivated by the recent paper by Cazenave et al. [1] which deals with the global existence and blow-up for the parabolic equation with non-local in time non-linearity

$$u_t - \Delta u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s) \mathrm{d}s \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.2}$$

where $0 \le \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^N)$, which is a particular case of (1.1) corresponding to $\beta = 2$. They proved that, if

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma)_+}$$

and

$$p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0, +\infty],$$

where $(u)_+ = \max(0,u)$, then:

(i) If $\gamma \ne 0$, $p \le p_*$, and $u_0 \ge 0$, $u_0 \not\equiv 0$, then u blows up in finite time.
(ii) If $\gamma \ne 0$, $p > p_*$, and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$(where $q_{sc} = N(p-1)/(4-2\gamma)$) with $\|u_0\|_{L^{q_{sc}}}$ sufficiently small, then u exists globally.

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If $\gamma = 0$ then all non-trivial positive solutions blow-up as proved by Souplet in [2]. Their study reveals the surprising fact that for Eq. (1.2) the critical exponent in Fujita's sense p_* is not the one predicted by scaling.

Fino and Kirane [3] explained this by the fact that their equation can be formally converted into

$$D^{\alpha}_{0|t} u_t - D^{\alpha}_{0|t} \Delta u = |u|^{p-1} u, \tag{1.3}$$

where $D^{\alpha}_{0|t}$ is the left-sided Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$ defined by: if $AC[0, T]$ is the space of absolutely continuous functions on $[0, T]$ with $0 < T < \infty$, then, for $f \in AC[0, T]$, the left-handed Riemann–Liouville fractional derivative $D^{\alpha}_{0|t} f(t)$ of order $\alpha \in (0, 1)$ is defined by (see [4])

$$D^{\alpha}_{0|t} f(t) := D J^{1-\alpha}_{0|t} f(t),$$

for all $t \in [0, T]$, where $D := \mathrm{d}/(\mathrm{d}t)$ is the usual derivative, and

$$J^{\alpha}_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s$$

is the Riemann–Liouville fractional integral defined in [4], where $\alpha = 1-\gamma \in (0, 1)$. Eq. (1.3) is a pseudo-parabolic equation and as it is well known scaling is efficient for detecting the Fujita exponent only for equations of parabolic type.

Needless to say that the equation considered by Cazenave et al. [1] is a genuine extension of the one considered by Fujita in his pioneering work [5].

In their article, concerning blowing-up solutions, Fino and Kirane [3] present a different proof from the one presented in [1], and for the more general Eq. (1.1) they have shown that their proof is more versatile and can be applied to more non-linear equations. They have shown that:

(1) For $u_0 \geq 0$, $u_0 \not\equiv 0$, and $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, if

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N - \beta + \beta\gamma)_+} \quad \text{or } p < \frac{1}{\gamma},$$

then all solutions of problem (1.1) blow-up in finite time.

(2) For $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, where $p_{sc} := N(p-1)/\beta(2-\gamma)$, if

$$p > \max\left\{1 + \frac{\beta(2-\gamma)}{(N - \beta + \beta\gamma)_+}; \frac{1}{\gamma}\right\},$$

and $\|u_0\|_{L^{p_{sc}}}$ sufficiently small, then u exists globally.

The method used to prove the blow-up result is the test function method considered by Mitidieri and Pohozaev [6,7] and Kirane et al. [8,9]. Furthermore, in the case $\beta = 2$, they derive the blow-up rate estimates for the parabolic Eq. (1.1) and have shown that:

If u is the blowing-up solution of (1.1) at the finite time $T^* > 0$, then there are constants $c, C > 0$ such that $c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(., t) \leq C(T^* - t)^{-\alpha_1}$ for $1 < p \leq 1 + 2(2-\gamma)/(N - 2 + 2\gamma)_+$ or $1 < p < 1/\gamma$ and all $t \in (0, T^*)$, where $\alpha_1 := (2-\gamma)/(p-1)$.

An upper bound of the blow-up time is always obtained via the local existence theorem. Here, we present a lower bound of the blow-up time following an idea of Payne [10].

2. Lower bound of the blow-up time

Theorem 2.1. *Let T^* be the blow-up time of the blowing-up solution to Eq. (1.1). We have the lower estimate of T^**

$$\frac{(1-\gamma)\Gamma(1-\gamma)}{(p-1)} \frac{1}{u_*^{p-1}(0)} \leq T^{*(2-\gamma)},$$

where $u_*(0) = \sup_x u_0(x)$.

Proof. Set $\Psi_k(t) = \int_{\mathbb{R}^N} u^{2k} \mathrm{d}x$. Then

$$\begin{aligned}
\Psi_k'(t) &= 2k \int_{\mathbb{R}^N} u^{2k-1} u_t \mathrm{d}x \\
&= -2k \int_{\mathbb{R}^N} u^{2k-1} (-\Delta)^{\frac{\beta}{2}} u \mathrm{d}x + \frac{2k}{\Gamma(1-\gamma)} \int_{\mathbb{R}^N} u^{2k-1} \left(\int_0^t (t-s)^{-\gamma} u^p \mathrm{d}s \right) \mathrm{d}x \\
&\leq \frac{2k}{\Gamma(1-\gamma)} \int_{\mathbb{R}^N} u^{2k-1} \left(\int_0^t (t-s)^{-\gamma} u^p \mathrm{d}s \right) \mathrm{d}x.
\end{aligned}$$

Now let $u_*(t) = \sup_x u(t,x)$. Then we have

$$\Psi_k'(t) \le \frac{2k}{(1-\gamma)\Gamma(1-\gamma)} u_*^{p-1}(t) \int_{\mathbb{R}^N} u^{2k}\, t^{1-\gamma}\, \mathrm{d}x$$

or upon integration we obtain

$$\Psi_k(t) \le \Psi_k(0) \exp\left\{ \frac{2k}{(1-\gamma)\Gamma(1-\gamma)} \int_0^t u_*^{p-1}(\eta)\eta^{1-\gamma}\mathrm{d}\eta \right\}$$

or

$$u_*(t) \le u_*(0) \exp\left\{ \frac{t^{1-\gamma}}{(1-\gamma)(\Gamma(1-\gamma))} \int_0^t u_*^{p-1}(\eta)\mathrm{d}\eta \right\}. \tag{2.1}$$

Since the solution blows-up as $t \to T^*$, hence $u_*(t) \to +\infty$ as $t \to T^*$, and further $\int_0^{T^*} u_*^{p-1}(\eta)\mathrm{d}\eta = +\infty$. Using inequality (2.1), we obtain

$$-\exp\left\{ -\frac{(p-1)T^{*(1-\gamma)}}{(1-\gamma)\Gamma(1-\gamma)} \int_0^t u_*^{p-1}(\eta)\mathrm{d}\eta \right\} \le \frac{(p-1)}{(1-\gamma)\Gamma(1-\gamma)} u_*^{p-1}(0)$$

and differentiating the left hand side of the last inequality with respect to the variable t, using inequality (2.1) and an integration leads to

$$1-\exp\left\{ -\frac{(p-1)T^{*(1-\gamma)}}{(1-\gamma)\Gamma(1-\gamma)} \int_0^{T^*} u_*^{p-1}(\eta)\mathrm{d}\eta \right\} \le \frac{T^{*(2-\gamma)}(p-1)}{(1-\gamma)\Gamma(1-\gamma)} u_*^{p-1}(0).$$

Since $\int_0^{T^*} u_*^{p-1}(\eta)\mathrm{d}\eta = +\infty$, and hence

$$\exp\left\{ -\frac{(p-1)T^{*(1-\gamma)}}{(1-\gamma)\Gamma(1-\gamma)} \int_0^{T^*} u_*^{p-1}(\eta)\mathrm{d}\eta \right\} = 0,$$

it follows then that a lower bound for T^* is given by

$$\frac{(1-\gamma)\Gamma(1-\gamma)}{(p-1)} \frac{1}{u_*^{p-1}(0)} \le T^{*(2-\gamma)}$$

which concludes the proof. □

3. On the Lemma 2.1 of [1]

In [1], the following lemma has been used; no proof of it was given.
Finding the hypotheses not precise, we decided to give a proof of it which seems instructive to us.

Lemma 3.1 (*[1]*). *Let $a > 0$, $b > 0$, $p > 1$, and $v(t)$ be a positive solution of the equation:*

$$v''(t) + av'(t) = bv^p(t). \tag{3.1}$$

Given any $T > 0$, there exists a constant $K = K(a,b,p,T) > 0$ such that if $v(0) > K$ and $v'(0) > -av(0)$, then v blows up in a finite time $T_{\max} \le T$.

Proof. We will give a proof based on the test function method; this proof is simple and gives accurate results.
Let $u \ge 0$ be a solution of Eq. (3.1) and $\varphi(t) = \left(1 - \frac{t}{T}\right)_+^2$. Then

$$\int_0^T v''(t)\varphi(t)\mathrm{d}t = -v'(0) - \frac{2}{T}v(0) + \int_0^T v(t)\varphi''(t)\mathrm{d}t.$$

Also

$$\int_0^T v'(t)\varphi(t)\mathrm{d}t = -v(0) - \int_0^T v(t)\varphi'(t)\mathrm{d}t$$

whereupon:

$$b\int_0^T v^p(t)\varphi(t)\mathrm{d}t = -\left(\left(\frac{2}{T}+a\right)v(0) + v'(0)\right) - \int_0^T v(t)\varphi'(t)\mathrm{d}t + \int_0^T v(t)\varphi''(t)\mathrm{d}t.$$

Now, let $\Lambda = v'(0) + \left(a + \frac{2}{T}\right) v(0) > 0$, then

$$\Lambda + b\int_0^T v^p(t)\varphi(t)\mathrm{d}t \le \varepsilon \int_0^T v^p(t)\varphi(t)\mathrm{d}t + C_\varepsilon \left(\int_0^T \varphi^{-p'/p}(t)|\varphi''(t)|^{p'}\mathrm{d}t + \int_0^T \varphi^{-p'/p}(t)|\varphi'(t)|^{p'}\mathrm{d}t\right)$$

where $p + p' = pp'$.

Choosing $\varepsilon = b/2$ for example, we obtain

$$\Lambda + \frac{b}{2}\int_0^T v^p(t)\varphi(t)\mathrm{d}t \le C(b)\left(\int_0^T \varphi^{-p'/p}(t)[|\varphi''(t)|^{p'} + |\varphi'(t)|^{p'}]\mathrm{d}t\right). \tag{3.2}$$

Now, choosing the scaling $t = T\tau$ we obtain the estimates

$$\int_0^T \varphi^{-p'/p}(t)|\varphi''(t)|^{p'}\mathrm{d}t \le CT^{-2p'+1}; \qquad \int_0^T \varphi^{-p'/p}(t)|\varphi'(t)|^{p'}\mathrm{d}t \le CT^{-p'+1}. \tag{3.3}$$

As

$$-p' + 1 < 0 \Longrightarrow -2p' + 1 < 0,$$

in view of (3.3), the right-hand side of (3.2) goes to zero when $T \longrightarrow +\infty$.

This is a contradiction because $\Lambda > 0$. $\square$

Remark 3.2. (1) The requirement $\Lambda = v'(0) + \left(a + \frac{2}{T}\right) v(0) > 0$ is weaker than the one given in [1].

(2) In [1], nothing is said about the constant K nor about T the bound of $T_{\max}$.

It is clear from

$$\Lambda + \frac{b}{2}\int_0^T v^p(t)\varphi(t)\mathrm{d}t \le C(b)\left(\int_0^T \varphi^{-p'/p}(t)[|\varphi''(t)|^{p'} + |\varphi'(t)|^{p'}]\mathrm{d}t\right)$$

that

$$\Lambda \le C(b)\{T^{-2p'+1} + T^{-p'+1}\} \tag{3.4}$$

and

$$\frac{1}{\Lambda} \le \frac{1}{v'(0)}.$$

To obtain a bound on $T_{\max}$, let us distinguish the following two cases in (3.4):

(3) $T > 1 \Rightarrow T^{-2p'+1} < T^{-p'+1}$ and hence

$$T \le \left(\frac{2C(b)}{\Lambda}\right)^{\frac{1}{p'-1}} \le \left(\frac{2C(b)}{v'(0)}\right)^{\frac{1}{p'-1}} = \left(\frac{2C(b)}{v'(0)}\right)^{p-1}.$$

(4) $T < 1 \Rightarrow T^{-2p'+1} > T^{-p'+1}$ and hence

$$T \le \left(\frac{2C(b)}{\Lambda}\right)^{\frac{1}{2p'-1}} \le \left(\frac{2C(b)}{v'(0)}\right)^{\frac{1}{2p'-1}} = \left(\frac{2C(b)}{v'(0)}\right)^{\frac{p-1}{p+1}}.$$

4. Conclusion

The paper gives a simple and versatile proof of a key lemma in [1], and a lower bound on the blow-up time following Payne's idea [10].

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